GENERATING PLANAR 4-CONNECTED GRAPHS

BY

DAVID BARNETTE[†]

ABSTRACT

In this paper, we introduce three operations on planar graphs that we call face splitting, double face splitting, and subdivision of hexagons. We show that the duals of the planar 4-connected graphs can be generated from the graph of the cube by these three operations. That is, given any graph G that is the dual of a planar 4-connected graph, there is a sequence of duals of planar 4-connected graphs $G_0, G_1, \ldots, G_n$ such that G_0 is the graph of the cube, $G_n = G$, and each graph is obtained from its predecessor by one of our three operations.

1. Introduction

It is a well known theorem that the planar 3-connected graphs can be generated from the graph of the tetrahedron by a process known as face splitting (see [2] and [3] for related results). In this paper we investigate the problem of generating the planar 4-connected graphs. Our main theorem is that the duals of the planar 4-connected graphs can be generated from the graph of the cube using three operations; that is, given any graph G that is the dual of a planar 4-connected graph, there is a sequence $G_1, G_2, \cdots, G_n$ such that G_1 is the graph of the cube, $G_n = G$, G_i is the dual of a planar 4-connected graph and can be obtained from G_{i-1} by one of our three operations.

2. Preliminary Definitions

Since all graphs in this paper are planar, we shall omit the word planar from here on. We shall use the terms isomorphic and homeomorphic when speaking of graphs. Two graphs are *isomorphic* provided

† Research supported by a Sloan Foundation fellowship and by NSF Grant # GP-27963.

Received June 23, 1972

there is a one-to-one correspondence of the vertices of one to the vertices of the other such that two vertices determine an edge of one if and only if the corresponding vertices in the other graph determine an edge. When we speak of two graphs being *homeomorphic*, we mean that they are homeomorphic in the topological sense, thus homeomorphism is weaker than isomorphism.

A *path* in a graph is a sequence of vertices $v_1, v_2, \cdots, v_n$ such that for all $1 \leqq i \leqq n-1$, v_i and v_{i+1} determine an edge of the graph and no vertex appears twice in the sequence. If the set of vertices of G_2 is a subset of the set of vertices of G_1 then $G_1 \sim G_2$ is the graph consisting of the edges in G_1 but not in G_2 and their vertices. If V is a set of vertices in G then $G \sim V$ is the subgraph of G determined by the vertices of G that are not in V.

If H is a subgraph of G then the *complement* of H (in G) is denoted by $\sim H$ and is the graph $G \sim H$. The *vertices of attachment* of H are the vertices belonging to both H and $\sim H$.

A graph G_2 is a *refinement* of G_1 provided G_2 can be obtained from G_1 by replacing some of the edges of G_1 by paths. A vertex of G is a *major vertex* provided it has valence at least three; it is a *minor vertex* if it has valence two. A graph G_2 is a *contraction* of G_1 provided G_1 and G_2 are homeomorphic and G_2 contains no minor vertices. An *arc* of a graph G is a subgraph of G that is homeomorphic to a segment whose endpoints are major vertices of G, and whose other vertices (if any) are minor vertices of G. If G_1 and G_2 are two graphs then by $G_1 \cup G_2$ we mean the graph whose vertex set is the union of the vertex sets of G_1 and G_2 and whose edges are the edges of G_1 or G_2.

We shall denote the edge with vertices v_1 and v_2 by v_1v_2. If v_1 and v_2 are vertices of an arc of G, then v_1v_2 denotes the path in the arc from v_1 to v_2. The arc v_1v_2 will always be well defined in the graphs in this paper.

A graph G is *n-connected* provided that between any two vertices of G there are n independent paths (that is, paths that meet only at their endpoints). Equivalently, a graph is n-connected provided that it has at least $n+1$ vertices and cannot be disconnected by removing fewer than n vertices.

If G is embedded in the plane Π, then the *faces* of G are the subgraphs of G that bound the connected components of $\Pi \sim G$.

By an *n-cycle* in G we mean a sequence $F_1, F_2, \cdots, F_n$, $n \geqq 3$, of faces of G such that for $1 \leqq i \leqq n-1$, F_i and F_{i+1} have a common vertex and F_n and F_1

have a common vertex. An n-cycle is *nontrivial* provided no vertex belongs to more than two faces in the cycle.

If G is embedded in the plane we can construct a graph G^*, called the *dual* of G, by placing a vertex of G^* in each face of G and joining two vertices of G^* if and only if the corresponding faces meet on an edge.

We shall define a graph to be *nice* if its contraction is the dual of a 4-connected graph.

We shall say that a graph G_2 is obtained from a graph G_1 by *splitting* face F of G_1 provided we can get G_2 by adding an edge across face F. There are three ways this can be done (see Fig. 1) depending on how many new vertices are introduced by the splitting.

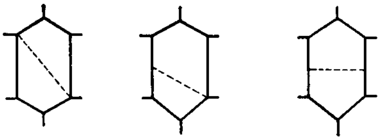

Fig. 1.

We shall say that G_2 is obtained from G_1 by *subdividing a hexagon* F of G_1 provided G_2 is obtained by adding a vertex and three edges to a six-sided face F of G_1 as shown in Fig. 2.

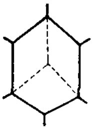

Fig. 2.

Our third operation is called double face splitting. It consists of splitting two quadrilateral faces meeting at two 3-valent vertices as illustrated in Fig. 3.

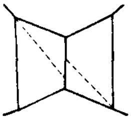

Fig. 3.

If F is a face of G with three major vertices, we shall call F a *triangular* face. If F has four major vertices, we shall call F a *quadrilateral* face. *Pentagonal, hexagonal* faces, etc., will be defined similarly.

3. Preliminary lemmas

LEMMA 1. *A 3-connected graph remains connected when we remove the vertices of one face of the graph.*

PROOF. By a theorem of Steinitz [2] every planar 3-connected graph is the graph of a 3-dimensional convex polytope. It follows from a theorem of Balinski [1] that the graph G of a 3-dimensional convex polytope cannot be disconnected by removing vertices of just one face of G. Since faces of the polytope correspond to faces of the graph the proof is complete.

LEMMA 2. *The dual of a 3-connected graph is 3-connected.*

This is a well-known theorem that may be deduced from Steinitz' theorem.

LEMMA 3. *Suppose the vertices v_1, v_2 and v_3 separate a 3-connected graph G. Then there is a 3-cycle F_1, F_2, F_3 such that v_1 belongs to F_1 and F_2, v_2 belongs to F_2 and F_3, and v_3 belongs to F_3 and F_1.*

PROOF. Let $F^1, F^2, \cdots, F^n$ be the faces of G that contain v_1. The set of edges of $F^1, \cdots, F^n$ that do not meet v_1 will form a simple circuit C that meets every connected component of $G \sim \{v_1, v_2, v_3\}$, for if C missed one of the components then $\{v_2, v_3\}$ would separate G, contradicting the 3-connectedness of G. In order to separate C, the vertices v_2 and v_3 must lie on C.

The vertices v_2 and v_3 cannot lie on the same face of the sequence $F^1, \cdots, F^k$ because this would imply that we could disconnect G by removing vertices of one face of G.

We may now choose two faces of $F^1, \cdots, F^k$, one containing v_1 and v_2 and one containing v_1 and v_3. By repeating this argument using faces surrounding v_2 we can get a face containing v_2 and v_3.

Now we shall characterize the duals of 4-connected graphs.

LEMMA 4. *G^* is the dual of a 4-connected graph G if and only if G^* is 3-connected and has no nontrivial 3-cycles.*

PROOF. Suppose F_1, F_2, F_3 is a 3-cycle in a graph G with vertices v_1, v_2 and v_3 such that v_1 belongs to F_1 and F_2, v_2 belongs to F_2 and F_3 ,and v_3 belongs to F_3 and F_1. In the dual G^*, corresponding to v_1, v_2 and v_3 will be faces F_1^*, F_2^* and F_3^*, and corresponding to F_1, F_2 and F_3 will be vertices v_1^*, v_2^*, v_3^*, with v_1^* belonging to F_3^* and F_1^*, v_2^* belonging to F_1^* and F_2^*, and v_3^* belonging to F_2^* and F_3^*. Thus we see that a 3-cycle in a graph corresponds to a 3-cycle in the dual.

Suppose G is 4-connected. Then the only 3-cycles that G could have would be those surrounding triangular faces of G, for otherwise the 3-cycle would provide us with three vertices that separate the graph. This implies that in G^* the only 3-cycles are trivial ones that surround 3-valent vertices. It also follows that G^* is 3-connected because it is the dual of 3-connected graph.

Suppose now, that G^* has no nontrivial 3-cycles and is 3-connected. This implies that G is 3-connected and the only 3-cycles in F are those surrounding triangular faces. If G were not 4-connected then it could be disconnected by removing three vertices v_1, v_2 and v_3. Thus there is a 3-cycle F_1, F_2, F_3 such that v_1 belongs to F_1 and F_2, v_2 belongs to F_2 and F_3, and v_3 belongs to F_3 and F_1. This implies that v_1, v_2, and v_3 are vertices of a triangular face of G, contradicting the fact that v_1, v_2, and v_3 separate G.

LEMMA 5. *If G_1 is the dual of a 4-connected graph and G_2 is obtained from G_1 by splitting a face of G_1 in such a way that G_2 has no triangular faces, then G_2 is the dual of a 4-connected graph.*

PROOF. One can easily show that the only nontrivial 3-cycle that can be created by face splitting in G_1 is a 3-cycle surrounding a triangular face, thus the lemma follows from Lemma 4.

LEMMA 6. *If G_1 is the dual of a 4-connected graph and G_2 is obtained from G_1 by subdividing hexagon F of G_1, then G_2 is the dual of a 4-connected graph.*

PROOF. No nontrivial 3-cycle in G_2 can contain two or three faces of G_2 inside F. If a nontrivial 3-cycle contained one face of G_2 in F, then replacing that face by F would give a nontrivial 3-cycle in G_1. If a nontrivial 3-cycle in G_2 contains no face of G_2 in F, then this is also a nontrivial 3-cycle in G_1.

LEMMA 7. *If G_1 is the dual of a 4-connected graph and G_2 is obtained from G_1 by double face splitting, then G_2 is the dual of a 4-connected graph.*

PROOF. Let us add the two edges to G_1 one at a time. Adding the first edge creates one nontrivial 3-cycle. Adding the second edge destroys the 3-cycle and does not create any new nontrivial 3-cycles, thus by Lemma 4 the proof is complete.

LEMMA 8. *The dual G^* of a 4-connected graph G contains a refinement of the graph of the cube.*

PROOF. Let $F^1, \cdots, F^k$ be the faces of G^* that meet some face F of G^*. Let

$F^1, \cdots, F^k$, be their cyclic order around F. Since G^* contains no nontrivial 3-cycles it follows that any two non-consecutive faces in $F^1 \cdots, F^k$, meet only on vertices of F. If we now take the set of edges that are edges of the F^i's but miss F, we have a simple circuit C missing F.

The face F has at least four vertices, thus we can get a refinement of the graph of the cube by taking F, C, and four disjoint edges of G^* joining F and C.

LEMMA 9. *If H is a subgraph of the dual of a 4-connected graph G and if H_1 is a subgraph of H with three vertices of attachment in H, then there is a path in G joining H to $\sim H$ and missing the vertices of attachment, unless H_1 consists of a vertex and the three edges meeting it.*

PROOF. If all paths in G from H to $\sim H$ passed through the vertices of attachment, then these vertices would separate G. These three vertices would then determine a nontrivial 3-cycle in G unless H_1 consists of a single vertex and the three edges of G meeting it.

4. The main theorem

THEOREM 1. *If G is the dual of a 4-connected graph then G can be generated from the graph of the cube by face splitting, double face splitting and subdivision of hexagons.*

PROOF. Our proof is by induction on the number of edges of G. The theorem is clearly true when $e = 12$. Suppose that the theorem is true for all graphs with fewer than n edges and suppose G has n edges ($n > 12$).

There is a second inductive argument in our proof. We shall now show that for every integer k, $12 \leqq k \leqq n$, there exists a nice subgraph of G with at least k arcs whose contraction can be generated from the graph of the cube by face splitting, double face splitting, and subdivision of hexagons. This we shall prove by induction on k. Proving this clearly will complete the inductive step of our first inductive argument. Our second induction is started by Lemma 8 when $k=12$.

Suppose that $12 < k \leqq n$. Let H_k be the subgraph of G guaranteed by the second inductive hypothesis and let G_k be its contraction. If H_k has more than k arcs we are done thus we assume H_k has exactly k arcs.

Our proof now consists of two parts, the first part being the case where H_k contains minor vertices.

Let A_1 be an arc in H_k containing a minor vertex. There must be a path in

$G \sim H_k$ joining some interior vertex of A_1 to some arc of H_k other than the arc A_1, for if not then we could disconnect G by removing the endpoints of A_1. Let v_1 and v_2 be the endpoints of Γ with v_1 on A_1 and v_2 on another arc A_2.

If adding Γ to H_k does not create any triangular faces then the contraction of $H_k \cup \Gamma$ is the desired graph. That is the contraction of $H_k \cup \Gamma$ is the dual of a 4-connected graph (by Lemma 5), it is homeomorphic to a subgraph of G (namely $H_k \cup \Gamma$), it can be generated from the graph of the cube by our operations, and $H_k \cup \Gamma$ has at least $k + 1$ arcs.

Suppose now that any arc we add to H_k with endpoints on distinct arcs creates a triangular face and suppose v_1v_2 is such an arc with v_1 on arc A_1 of H_k and v_2 on arc A_2 of H_k, with A_1 and A_2 meeting at v_3. If we remove v_1v_3 from $H_k \cup v_1v_2$, we obtain a graph whose contraction is 3-connected and has no nontrivial 3-cycles (we leave this to the reader to verify), thus we will have a good graph. If v_3 has valence greater than three in H_k then our new graph has more arcs than H_k and fewer edges than G. It follows by induction that the contraction of this new graph can be generated by our operations and we are done.

From here on we shall assume that any arc that we may add with endpoints on distinct arcs will have its endpoints on two arcs of H_k that meet at a vertex that is 3-valent in H_k.

We now consider several cases.

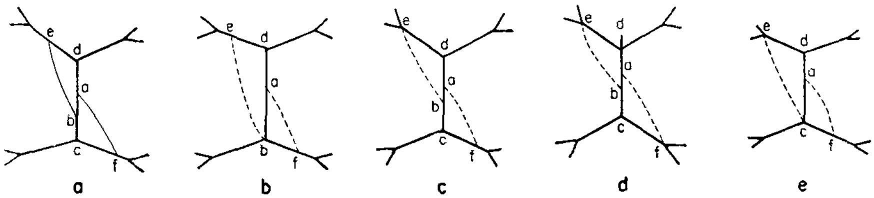

Fig. 4.

Case I. We may add two arcs to H_k as illustrated in Fig. 4. In Figs. 4a, 4b, and 4e we may remove *cf*, obtaining a good graph which by induction can be generated by our operations. In Fig. 4c we may remove *ed* obtaining a good graph which by induction can be generated by our operations. In Fig. 4d we may remove *cf* unless c is 3-valent and the face of H_k containing d, c and e is a quadrilateral. In this case we may remove *ed* unless d is 3-valent and the face of H_k containing d, c and f is a quadrilateral. But if this is also true then $H_k \cup eb \cup af$ is obtained from H_k by double facet splitting. In each case we have created a nice subgraph of G with more than k arcs that can be generated by our operations.

From here on we shall assume that we cannot add an arc to H_k without producing a triangle and that we cannot add two arcs to H_k as illustrated in Fig. 4.

Case II. We may add an arc to H_k as illustrated in Fig. 5. We shall assume that among all graphs isomorphic to $H_k \cup af$ that this one encloses the largest number of faces of G in the face F of $H_k \cup ac$ determined by a, f and c.

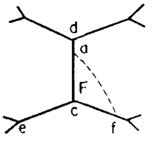

Fig. 5.

Now consider the subgraph E of G consisting of F and everything inside F. By Lemma 8 there must be some arc gh with g on one of the arcs af, fc or ac, with h not in E and with the arc missing a, f and c. Figure 6 shows five of the six possible positions of the arcs.

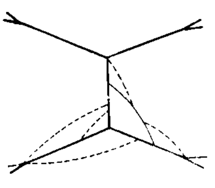

Fig. 6.

The sixth possible position is illustrated in Fig. 7 and is only possible when the face of H_k containing d, f and c is a quadrilateral. In this case we remove dg from $H_k \cup af \cup gh$ producing a good graph whose contraction, by induction, can be generated by our operations.

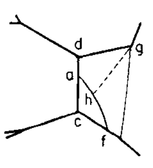

Fig. 7.

All other positions are ruled out by the maximality of F or by our assumption that certain arcs or combinations of arcs do not exist.

Returning to the other five ways of adding arcs, we observe that if c has valence greater than 3 and if we can add an arc as in Fig. 8a or 8b, then we can remove bc to obtain a good graph which by induction can be generated by our operations. From here on we shall assume that an arc can be added as in Fig. 8 only if c is 3-valent. Note that from here on we cannot add an arc as in Fig. 8b because of the maximality of F. We now add one of each of the remaining four types of

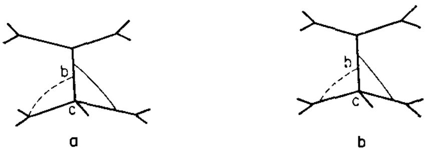

Fig. 8.

arcs in Fig. 6 in G. We shall call the new graph H_k'. Taking symmetry into account, we have eight different possibilities depending on how many such arcs are present. These are illustrated in Fig. 9. The heavy lines are for $H_k \cup ac$. The lighter lines indicate the new arcs that may be added. In each case we have a subgraph E' of H_k' with three vertices of attachment x, y and z. By Lemma 9, there is an arc from E' to its complement missing x, y and z.

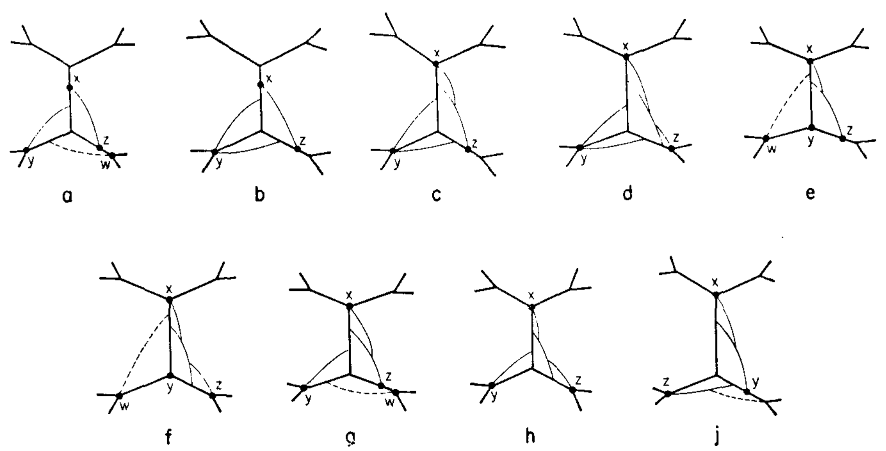

Fig. 9.

The reader may check that in Figs. 9b, 9c, and 9h, no such arc may be added without violating the maximality of F or the non-existence of certain arcs or combinations of arcs. In the remaining figures, there is one kind of arc which may be added. These arcs are indicated by dashed lines. In this case we add the arc and we have a new subgraph E'' which must admit a path from E'' to its complement, missing its vertices of attachment. The reader may check that no other such arc can be added.

Case III. We cannot add an arc as in Case II, but we can add an arc as illustrated in Fig. 10. Again we assume that the face F is maximal and we consider the subgraph E enclosed by this face and we determine the possible ways an arc may join E to its complement.

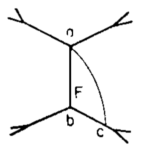

Fig. 10.

The case where an arc as illustrated in Fig. 11 can be added, can be taken care of as in Case I.

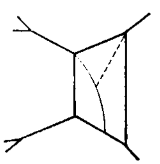

Fig. 11.

The other combinations of arcs that can be added are illustrated in Fig. 12.

Again, after adding these arcs, we have a subgraph E' such that an arc must join it to its complement and miss its vertices of attachment. In Fig. 12b, no such arc can be added. In Fig. 12a, one arc can be added as indicated, but after this arc is added we may argue as in Case II and conclude that no other arc may now be added.

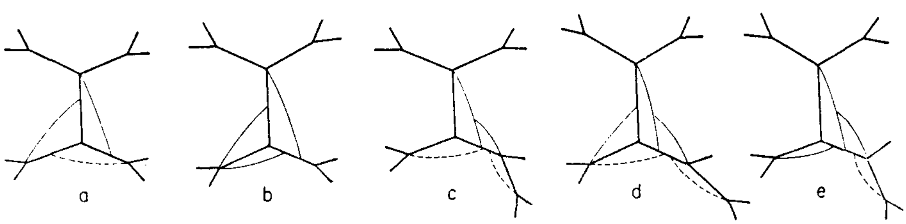

Fig. 12.

Before we do the cases in Figs. 12a 12d and 12e we shall treat the case where there exist two arcs as in Fig. 13. If we remove xy we produce a nice graph which by induction has a contraction that can be generated by our operations. From now on we may assume that there do not exist two arcs as in Fig. 13.

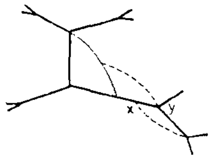

Fig. 13.

This now leaves us with only one type of arc that may be added in Figs. 12c and 12d and no arc that can be added in Fig. 12e. In Figs. 12c and 12d, we add the arc and then use the argument from Case II to conclude that we cannot add another one, getting a contradiction.

Case IV. There do not exist any arcs as in Cases II and III, but an arc as illustrated in Fig. 14 exists.

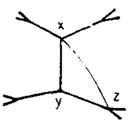

Fig. 14.

We argue as above. This time because we have ruled out so many types of arcs, there are only three types of arcs that can join E to its complement. These are illustrated in Fig. 15.

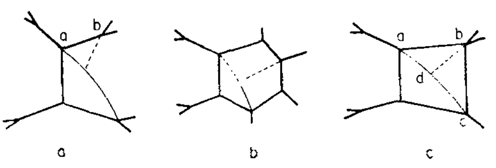

Fig. 15.

In Fig. 15a we remove ab producing a good graph and by induction we are done. In Fig. 15b, we have a graph obtained from H_k by subdividing a hexagon.

In Fig. 15c, the graph enclosed by ab, bc and ac will admit an arc joining it to its complement and missing its vertices of attachment. This can be done in essentially two ways (Fig. 16).

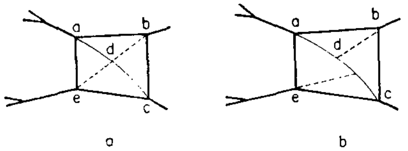

Fig. 16.

In Fig. 16b, we remove ab and ce producing a good graph, and by induction we are done. In Fig. 16a, the subgraph enclosed by ad, de and ae must admit an arc joining it to its complement and missing the vertices of attachment. This can be done in two ways (Fig. 17). In Fig. 17a, we produce the desired good graph by removing ab, ec and db. Now by induction we are done. In Fig. 17b, we remove ab, ec and dg and by induction we are done.

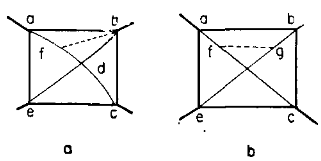

Fig. 17.

We now turn to the second half of our proof: the case where H_k contains no minor vertices. Among all subgraphs of G homeomorphic to H_k we shall assume that H_k has a maximum number of vertices.

Let v_1 be a vertex of H_k that meets an edge of G that is not in H_k. If we begin on this edge and travel along edges of G we eventually return to H_k. Furthermore, we will not return to an endpoint of an edge of H_k that meets v_1 (note that each arc of H_k is an edge of H_k) because we could replace the edge by our path and increase the number of vertices, contradicting the maximality of H_k. If we add an arc to H_k we will produce the desired graph unless adding the arc creates a triangular face (or possibly two triangular faces). This case, however, is disposed of by the argument of Case IV of the first part of our proof.

5. Remarks

One might ask whether all three of our operations are necessary, particularly double face splitting and subdividing hexagons. To see that double face splitting is necessary, observe that it is needed to produce the graph that one obtains by applying double face splitting to the graph of the cube. To see that subdivision of hexagons is necessary, observe that the graph of the rhombic dodecahedron, Fig. 18, can only be obtained from a smaller dual of a 4-connected graph subdividing a hexagon.

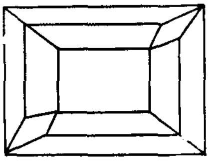

Fig. 18.

Generating the duals of the 4-connected graphs is equivalent to generating the 4-connected graphs. The reader may, if he wishes, restate the results in dual form. In this case the operations would be three types of "vertex splitting".

References

1. M. Balinski, *On the graph structure of convex polyhedra in n-space*, Pacific J. Math. **11** (1961), 431–434.
2. E. Steinitz and H. Rademacher, *Vorlesungen über die Theorie der Polyeder*, Berlin, 1934.
3. W. T. Tutte, *A theory of 3-connected graphs*, Indag. Math. **23** (1961) No. 4, 441–455.

University of California
Davis, California, U.S.A.